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Shock-Wave Structure in Porous Solids*

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In this paper several variations of a simple theory of dynamic compaction of porous solids are presented and discussed. This theory elaborates the conventional theory of shock propagation in such a way that the shock structures observed to propagate in these materials can be described. Steady-wave profiles are calculated for several compaction models, and the inference of constitutive equations from experimental data is discussed. It is shown that the theory can be made to reproduce steady-wave profiles observed in the usual plate-impact experiments exactly.

I. INTRODUCTION

Porous solids occur in such forms as geologic materials, manufactured foams, powder-metal compacts, and ceramics. During the past decade a substantial effort has been directed toward achieving an understanding of the propagation of moderate-amplitude plane waves of uniaxial strain in these materials.[1-8] While both compressive-loading and release waves have been studied, the loading behavior has received the most attention because it is more easily investigated experimentally. From an analytical standpoint, it seems clear that the compression problem is the more difficult because unloading from partially compacted states involves only a small volume recovery along a relatively straight stress-strain path.[6,8]

For most analyses of compaction-wave propagation in these materials it has been assumed that the state of stress in the material, as it is being compacted, depends only on the state of strain. Hydrodynamic, elastoplastic, and elastic-locking models have been used extensively. In each of these models steady compaction waves are found to propagate in the form of one or more shocks. There has been some work directed toward more complicated models involving characteristic lengths[9,10] or times[11,12] but these theories are less developed. Historically, experiments have been conducted at the very high pressures induced by explosive detonation (see, for example, Refs. 13–15) and have been interpreted in terms of the Rankine-Hugoniot theory of shock propagation. The profiles of these high-amplitude waves are satisfactorily approximated as shocks because the actual wave thickness is small compared to propagation distances of interest. More recently, plate-impact experiments have been performed at pressures only moderately in excess of the static compaction threshold. The waveforms observed in these experiments are only crudely described as shocks because of the large amount of dispersion present. As an illustration of the sort of effects observed, we present the experimental records of Fig. 1(a) showing the profiles to which shocks of various amplitudes have evolved after propagating for a distance of 1.25 mm in porous iron samples. A plot of wave thickness as a function of stress amplitude is given in Fig. 1(b).

It is to describe these low-pressure observations that an improved theory must be developed. While it is easy to conceive of a number of effects that would contribute to the observed dispersion, it seems probable that the most influential is the lag experienced by the material in coming to equilibrium under load because of the time required for pore collapse. This dispersive effect is counteracted by the tendency of propagating waves to evolve toward shocks due to the rapidly decreasing compressibility of the material as it is compacted. The suggestion is obvious that here, as in the case of gas dynamics,[16] observed wave thicknesses are a result of the balance struck between these two conflicting tendencies. The fact that the stronger waves rise much more quickly than the weaker ones indicates that the shock-formation tendency is beginning to predominate over the dispersive mechanisms at the higher stresses. Unique (for a given amplitude) stable wave profiles where the tendencies are in perfect balance so that the wave can propagate unchanged in form exist and have been observed experimentally in a number of porous materials.

The objective of this paper is the exploration of a range of possible variations of a simple compaction theory, and the effect of these variations on steady waveforms. Primary attention has been given the problem of inference of constitutive equations from experimental data. Extension of the theory to cover a broader range of effects such as unloading or thermal response or treatment of the evolution of a disturbance into a steady wave is possible, but it is not discussed here. In Sec. II a brief review of some relevant aspects of wave propagation is given to introduce the notation and provide ready reference for a few formulas. The constitutive equations of a simple compaction theory are discussed in Sec. III. Section IV is devoted to the solution of specific problems, Sec. V to the experimental determination of material constitution, and Sec. VI to a summary of the important findings.

II. THEORY

A. Kinematic and Dynamic Preliminaries

In this section we consider only problems involving uniaxial compaction. The motion of a material point initially residing at a place X in an inertial coordinate space but which, at some later time t, has been

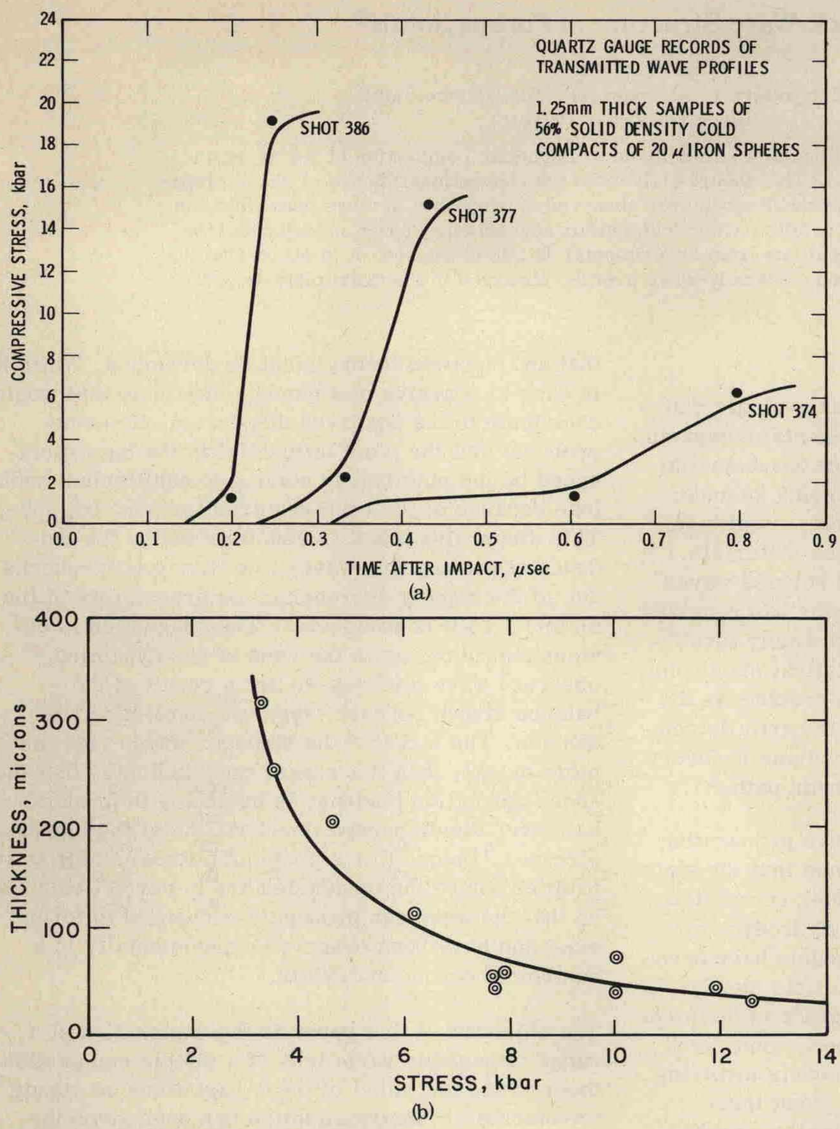

FIG. 1. Experimental results on shock compaction of porous iron. (a) shows profiles of stress waves of various amplitudes and (b) shows how the shock thickness varies with stress. These data have been communicated privately by Lysne and Halpin (Ref. 2).

displaced to a place x in this space may be expressed by an equation of the form $x = X + U(X, t)$. Strain (taken positive in compression), strain rate, and particle velocity are defined in terms of $U(X, t)$ by the relations

$$\epsilon = -U_X(X, t), \quad \dot{\epsilon} = -U_{Xt}(Xt), \quad u = U_t(x, t) \quad . \tag{1}$$

In order that mass be conserved locally, the specific volume $v(X, t)$ of the material point originally residing at X must be related to the (constant) initial specific volume of the body by the equation $v = v_0(1-\epsilon)$. The equation of motion that must be satisfied is

$$\sigma_X + \rho_0 U_{tt} = 0 \quad , \tag{2}$$

where $\sigma(X, t)$ is the normal stress (taken positive in compression) on planes $X =$ const. This equation of motion must be augmented by constitutive equations relating the dynamic variable σ to the kinematic variable U in order that specific problems may be solved. Before addressing this aspect of the problem it is convenient to make a few general remarks on steady-wave solutions of Eq. (2).

B. Steady Waves

In this section we consider the behavior of waves that propagate at constant velocity and unchanged in form, i.e., *steady waves*.[16-19] Any traveling-wave solution $f(y \pm ct)$ of the linear-wave equation $c^2\theta_{yy} = \theta_{tt}$ has this property; this equation involves neither dispersive tendency nor tendency toward shock formation, so perfect balance is achieved in any wave and it propagates unchanged in form. As mentioned in Sec. I, it is possible that these two effects could both be present and still allow certain waves to propagate steadily because of their counterbalancing tendencies. In Sec. IV we will see that such waves do exist within the scope of the theory presented here. They have been observed experimentally and have been proven to be the stable solution to which other waveforms evolve.

To study a wave propagating at the constant velocity V we introduce the coordinates

$$\xi = X - Vt \quad \text{and} \quad \tau = t \quad . \tag{3}$$

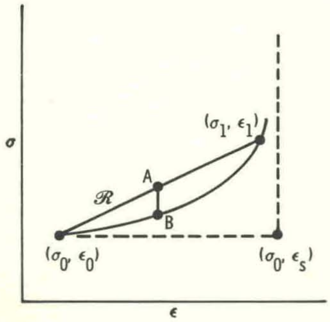

FIG. 2. Typical equilibrium stress-strain curve. Also shown is the Rayleigh line $\mathcal{R}$ for a wave taking the material from the state (σ_0, ϵ_0) to the state (σ_1, ϵ_1). The dashed lines represent the behavior of an ideal locking material.

In these coordinates Eq. (1) takes the form

$$\epsilon = -U_\xi, \quad \dot{\epsilon} = -U_{\xi\tau} + VU_{\xi\xi}, \quad u = U_\tau - VU_\xi \quad , \tag{4}$$

and Eq. (2) becomes

$$\sigma_\xi + \rho_0(U_{\tau\tau} - 2VU_{\tau\xi} + V^2U_{\xi\xi}) = 0 \quad . \tag{5}$$

In a steady wave propagating at the velocity V the field variables depend on ξ alone, so all τ derivatives vanish. Equation (4) then takes the form $\epsilon = -dU/d\xi$, $\dot{\epsilon} = Vd^2U/d\xi^2$, and $u = -V\,dU/d\xi$, leading to the important relations

$$\dot{\epsilon} = -V\frac{d\epsilon}{d\xi} \quad \text{and} \quad u = V\epsilon, \tag{6}$$

and Eq. (5) becomes $d(\sigma - \rho_0V^2\epsilon)/d\xi = 0$. This latter equation is readily integrated to give $\sigma - \rho_0 V^2\epsilon = \text{const}$. If the material is in a state σ_0, ϵ_0 at some point of the wave (actually we will assume this to be the case as $\xi \to \infty$), the constant can be evaluated and we have

$$\sigma - \sigma_0 = \rho_0V^2(\epsilon - \epsilon_0) \quad . \tag{7}$$

Since the second equation of (6) holds everywhere in the wave, it implies that $u_0 = V\epsilon_0$ and hence that

$$u - u_0 = V(\epsilon - \epsilon_0) \quad , \tag{8}$$

where u_0 is the particle velocity of the material in the state (σ_0, ϵ_0). Equations (7) and (8) together give

$$\rho_0(u - u_0)V - (\sigma - \sigma_0) = 0 \quad , \tag{9a}$$

$$(\epsilon - \epsilon_0)V - (u - u_0) = 0 \quad . \tag{9b}$$

These formulas are of the same form as the Rankine-Hugoniot shock equations. In particular, if u, ϵ, and σ in Eqs. (9) are assigned the values of these quantities behind the wave, then the two pairs of equations are identical when u_0, ϵ_0, and σ_0 refer to values of the unsubscripted variables ahead of the wave. This shows that any steady-wave experiment can be interpreted as a shock experiment if only equilibrium states behind the wave are of interest. As we shall see, the steady-wave analysis gives the complete wave profile. From Eq. (7) we see that the (σ, ϵ) path followed by a particle during the passage of a steady wave is the straight line, called the Rayleigh line $\mathcal{R}$, connecting the initial and final states in the (σ, ϵ) plane, and the wave speed is determined from the slope of this line:

$$\rho_0V^2 = (\sigma_1 - \sigma_0)(\epsilon_1 - \epsilon_0)^{-1} \quad . \tag{10}$$

In order that any existing steady-wave solutions be of practical interest, they must be stable and should also be unique. The heuristic discussion in Sec. I suggests that steady waves will be stable, and demonstrations of this stability in certain cases, as well as approximate solutions to wave evolution problems, have been given by Lighthill[16] and Bland.[19] One of the simpler theories to be discussed in this paper results in the equation $f(U_X)U_{XX} + \nu U_{XtX} = U_{tt}$ which has been studied rather extensively in a recent series of papers[20–22] in which the existence, uniqueness, and stability of steady-wave solutions are discussed.

The properties of steady waves discussed above, in addition to their simplicity, suggest their use in the experimental determination of constitutive equations. Unfortunately, as is apparent from the fact that each member of a broad class of disturbances evolves to the same steady wave, all information bearing on the evolutionary process is lost and one cannot expect steady-wave measurements alone to determine a constitutive equation uniquely. For this reason we must select a general class of constitutive equations as a starting point, and then demonstrate its applicability to the problem at hand. This is done in Secs. III–V.

III. CONSTITUTIVE EQUATIONS

As a starting point for the selection of a constitutive equation we note that experiments conducted on a variety of porous materials show that, for any given material, the states achieved as a result of dynamic compaction lie on a single, unique stress-strain curve $\sigma = \sigma_E(\epsilon)$ that is independent of the rate at which the compaction occurred. This observation suggests our first basic constitutive assumption: When the strain rate becomes zero at the end of a compaction process, the existing stress is a function of the strain, $\sigma = \sigma_E(\epsilon)$. We call this functional relationship an equilibrium stress-strain curve. These curves have been measured for a variety of materials, and several mathematical representations for them have been advanced. The most recent and complete theory of these curves known to us is that of Herrmann.[4]

The experiments that generate the equilibrium curves also show that, while a theory in which the stress is assumed to be a function solely of strain can predict the result of a compaction process correctly, it fails to provide an adequate description of the process itself. The fact that waves induced by planar impact do not propagate as centered simple waves, velocity discontinuities, or combinations thereof is the most typical indication of this failure. In order to obtain a theory capable of describing the

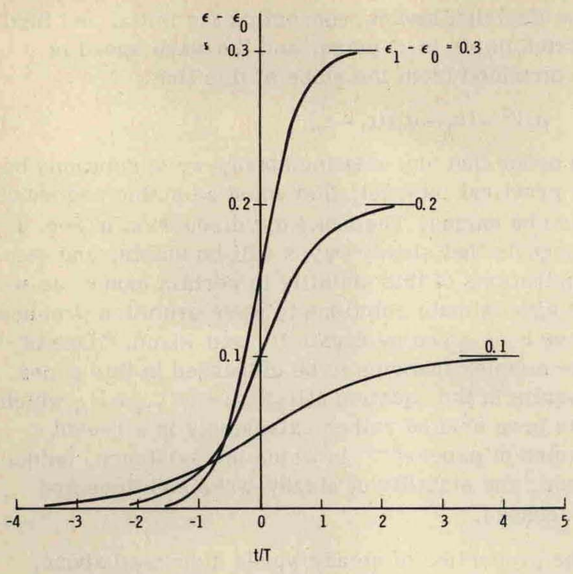

FIG. 3. Steady waveforms of various amplitudes in a material collapsing according to Eq. (20). We have taken $\beta^2 = 10$.

compaction process we generalize the rate-independent theory associated with the constitutive equation $\sigma = \sigma_E(\epsilon)$ by the inclusion of an additional contribution to the stress that is dependent on the rate of straining. This generalization is based on the discussion in the report of Johnson[11] and the paper of Butcher,[12] but focuses on the exploration of collapse phenomena while ignoring some of the range of effects covered in these articles.

Both Johnson and Butcher found it convenient to separate their thinking about material response into two parts dealing with the configuration of the material when loaded but in equilibrium (i.e., when the strain rate is zero) and the strain rate during collapse under applied load, respectively. While this has led to an unusual representation for a constitutive equation that is actually quite conventional, we have found the breakdown to be of great practical value and have continued to use it. The configuration of a body in equilibrium is, of course, obtained from the equilibrium stress-strain curve. The strain rate during a compaction process must be obtained from the complete rate-dependent constitutive equation. Since this equation is framed in a form inverted for strain rate, it is described as a "collapse rule". In the following we will show that the theory obtained in this way can be expressed in the conventional[23] form $\sigma = \sigma_E(\epsilon) + \psi(\epsilon, \dot{\epsilon})$, where $\psi(\epsilon, 0) = 0$.

The specific problem motivating this study lies in the calculations of Butcher which, while including a number of effects not included in this work and employing a very precise representation of the equilibrium stress-strain curve, fail to provide an adequate description of the observed steady-wave profiles in the material studied. The shortcomings seem traceable to the use of an oversimplified collapse rule [Eq. (1.19) of Ref. 12]. In this paper we generalize the linear collapse rule employed by Butcher in a way that seems plausible, fits conveniently into the conventional framework for continuum mechanics, and enables it to accommodate all steady-wave observations exactly.

A. Equilibrium Stress-Strain Curves

As noted previously, the determination of equilibrium stress-strain curves has been the object of many investigations over the past decade. The present work is built on this foundation and the equilibrium stress-strain curves called for in this paper are just those that have been determined before. Since these curves have fairly elaborate mathematical representations [often compounded by their expression in the form $\epsilon = f(\sigma)$], or exist only in graphical or tabular form, calculations using them are done by numerical means.

In all cases we have assumed that the equilibrium response of the material is described by a stress-strain curve that is concave toward the stress axis. For materials exhibiting a yield behavior, this requirement will be met only if the analysis is restricted to the range of states above some stress $\sigma_0 > 0$. Compaction waves propagating in a material having a yield point are unstable and separate into a low-amplitude precursor followed by the slower-propagating main compaction wave. When we take the stress σ_0 to be the precursor amplitude, then the present analysis is applicable to the description of the main compaction wave.

In order to accomplish the parameter studies of Sec. IV, it is convenient to have at hand a mathematical representation of the equilibrium stress-strain curve that, in addition to providing a reasonable approximation to real-material behavior at low strains, (a) is simple enough to allow analytical calculation of steady-wave forms for a variety of collapse rules, (b) involves a single dimensionless parameter that measures the departure from linearity, and (c) does not contribute to asymmetry of calculated steady waveforms. The function

$$\sigma = \sigma_0 + \rho_0 c_0^2 (\epsilon - \epsilon_0)[1 + \beta_2(\epsilon - \epsilon_0)] \tag{11}$$

fulfills these requirements; it approximates the observed behavior at low strains where dispersion effects are important and involves the parameter β characterizing the nonlinearity. It also eliminates the equilibrium behavior as a contributor to asymmetry of waveforms, but a discussion of what this means and a demonstration that it is accomplished must be postpone until Sec. IV. The extreme example of nonlinear equilibrium response is provided by the locking model represented by the dashed lines on Fig. 2 and discussed at the end of Sec. IV. The specific forms of the stress-strain curve given by Eq. (11) or by the locking model are not, of course, central features of the theory; they

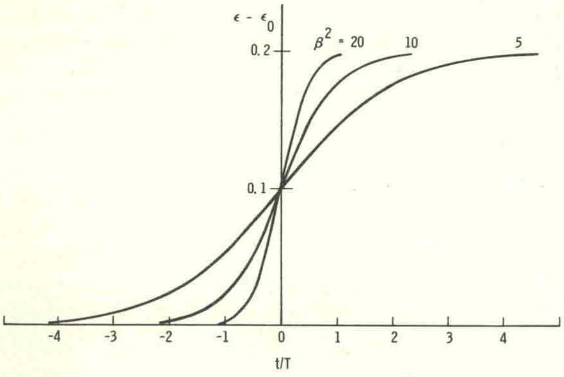

FIG. 4. Steady waveforms of amplitude $\epsilon_1 - \epsilon_0 = 0.2$ in a material collapsing according to Eq. (20). The effect of increasing curvature in the equilibrium stress-strain curve is illustrated.

are used merely to facilitate the parametric studies of Sec. IV.

B. Collapse Rules

In this section we address the question of how a porous solid collapses to an equilibrium state upon load application.

Let us consider a material point in a nonequilibrium state A and having the equilibrium stress-strain curve shown as the heavy solid line on Fig. 2. This figure has been drawn showing the state point A on the Rayleigh line, as it is in a steady wave, but we emphasize that the theory is intended to be applicable to any collapse process, and that A can refer to any state above the equilibrium curve. We will assume that the rate of collapse $\dot{\epsilon}$ of the material depends on its departure from equilibrium, and take as our measure of this departure the distance from A to the equilibrium state B at the same strain. With this assumption we have the collapse rule

$$\dot{\epsilon} = \phi_1(\sigma - \sigma_E(\epsilon)) \tag{12}$$

One generalization of this collapse rule that seems appropriate is the introduction of strain-dependent weighting in the calculation of the collapse rate associated with a given departure from equilibrium. In this case it would take the form

$$\dot{\epsilon} = \phi(\epsilon, \sigma - \sigma_E(\epsilon)). \tag{13}$$

The necessity for this generalization is suggested by observations to be discussed subsequently.

The simplest special case of Eq. (12) that may be of interest is that in which the function ϕ_1 is linear and homogeneous:

$$\phi_1 = (1/\sigma^* T)[\sigma - \sigma_E(\epsilon)] \quad , \tag{14}$$

the case considered by Butcher. Clearly this function must involve a characteristic time and a characteristic stress in order that the dimensions of each member of Eq. (12) be the same. For the same reason, Eq. (13) must also involve one or more pairs of such constants.

The quasistatic collapse, under constant applied stress, of a material governed by Eq. (14) is readily found to be given implicitly by

$$t/T = \int_{\epsilon_A}^{\epsilon} \{\sigma^* [\sigma_A - \sigma_E(\lambda)]^{-1}\} d\lambda \quad , \tag{15}$$

and hence to be dependent on the form of the function $\sigma_E(\epsilon)$ as well as the constant $T\sigma^*$.

Before discussing wave-propagation problems involving Eq. (13), let us consider what conditions must be satisfied by the function ϕ in order that the material respond in a plausible fashion. It is clear that the collapse rate must vanish when the material is in equilibrium, so we must have $\phi(\epsilon, 0) \equiv 0$. Similarly, it seems plausible that the collapse rate should increase for increasing departure from equilibrium at any given strain, so we require that ϕ be a monotonic increasing function of its second argument. Finally, our intent in including the explicit dependence of $\dot{\epsilon}$ on ϵ was to accomodate the possibility that, for a given departure from equilibrium, the collapse rate would be greater at large strains than at small strains because the reduced void size in the first instance would be expected to lead to a smaller effective characteristic collapse time. To achieve this objective we require that ϕ also be a monotonic increasing function of its first argument when the second is held fixed. Surfaces ϕ meeting the above conditions slope upward as one proceeds in the direction of either increasing strain or overstress. In the special case of Eq. (12) the surface becomes a cylindrical sheet with generators parallel to the ϵ axis, and when the collapse rule is linear this cylinder becomes a plane. In these latter cases, of course, the collapse rule is completely represented by a single-valued curve in the $(\dot{\epsilon}, \sigma - \sigma_E)$ plane.

To see that Eq. (13) fits naturally into the usual theory[23] of rate-dependent constitutive equations as stated above, we note that it can be rewritten in the form $\sigma = \sigma_E(\epsilon) + \psi(\epsilon, \dot{\epsilon})$, where $\psi(\epsilon, 0) \equiv 0$, and $\psi(\epsilon, \dot{\epsilon})\dot{\epsilon} \geq 0$ by invoking the monotonicity conditions just discussed. In this form we see that it exhibits the usual decomposition of stress into equilibrium and nonequilibrium parts. The special case of Eq. (12) has the simple inverted form $\sigma = \sigma_E(\epsilon) + \phi_1^{-1}(\dot{\epsilon})$, which, in the linear case considered by Butcher, is further simplified to $\sigma = \sigma_E(\epsilon) + \sigma^* T \dot{\epsilon}$.

IV. STEADY-WAVE PROFILES

In this section we consider steady-wave propagation in materials governed by the collapse rule of Eq. (13) and any equilibrium curve $\sigma = \sigma_E(\epsilon)$ that is concave to the stress axis in the region of interest. The solution for general forms of the collapse rule and equilibrium curve is reduced to quadrature, and explicit closed-form results are given for special cases.

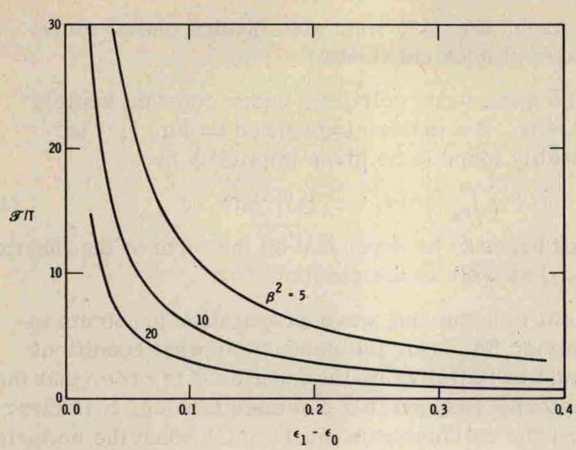

FIG. 5. Dependence of steady-wave rise time on amplitude according to Eq. (21).

Since the state far behind a propagating disturbance is one of equilibrium, $\sigma_1 = \sigma_E(\epsilon_1)$, we see from Eq. (10) that the speed of propagation of a steady wave is determined by the equilibrium stress-strain curve alone and is quite independent of the collapse rule. In general we have $V = \{(1/\rho_0)[\sigma_E(\epsilon_1) - \sigma_0](\epsilon_1 - \epsilon_0)^{-1}\}^{1/2}$. To solve for a steady waveform we simply substitute the first of Eqs. (6) and (7) into the collapse rule and integrate. In the general case we find that

$$\xi = -V \int_{(\epsilon_1+\epsilon_0)/2}^{\epsilon} \frac{d\lambda}{\phi(\lambda, \sigma_0 + \rho_0 V^2(\lambda - \epsilon_0) - \sigma_E(\lambda))}, \tag{16}$$

where the constant of integration has been chosen so that $\xi = 0$ at the half-amplitude point. Since $\phi(\epsilon, 0) = 0$ we see that the integrand is singular at both ϵ_0 and ϵ_1 so that the waveforms considered extend over the whole range $-\infty < \xi < \infty$. As a practical matter, however, we will see that the bulk of the variation is confined to a rather narrow region of space, or short interval of time. Since $\xi = X - Vt$, we can easily obtain the strain history at a fixed particle (in our examples we take $X = 0$; the waveform is independent of the choice). Stress and particle-velocity histories are obtained from the strain history by means of Eqs. (7) and (8) given in Sec. II.

In the special case where the stress-strain curve of Eq. (11) is used, Eq. (16) takes the simplified form

$$\xi = -V \int_{(\epsilon_1+\epsilon_0)/2}^{\epsilon} \frac{d\lambda}{\phi(\lambda, \rho_0 c_0^2 \beta^2 (\epsilon_1 - \lambda)(\lambda - \epsilon_0))}, \tag{17}$$

where we now have

$$V = c_0 [1 - \beta^2(\epsilon_1 - \epsilon_0)]^{1/2} . \tag{18}$$

A. Linear Collapse Rule

As a specific example, let us determine the steady-wave profile implied by Eq. (17) when the linear collapse function (14) is used. Evaluation of the integral is routine and yields the result

$$\xi = -\frac{VT}{\beta^2(\epsilon_1 - \epsilon_0)} \log_e \left(\frac{\epsilon - \epsilon_0}{\epsilon_1 - \epsilon}\right), \tag{19}$$

where V is given by Eq. (18) and where we have taken $\sigma^* = \rho_0 c_0^2$. The strain history obtained from Eq. (19) is

$$\epsilon(t) = \epsilon_0 + \frac{(\epsilon_1 - \epsilon_0)\exp[\beta^2(\epsilon_1 - \epsilon_0)t/T]}{1 + \exp[\beta^2(\epsilon_1 - \epsilon_0)t/T]} . \tag{20}$$

Graphs of these waveforms as functions of amplitude are shown on Fig. 3 for the case $\beta^2 = 10$ and on Fig. 4 for the fixed amplitude $\epsilon_1 - \epsilon_0 = 0.2$ and varying values of β^2. The stress and particle-velocity histories can be obtained from Eq. (20) through the simple algebraic relations (7) and (8). Examination of Eq. (20) (and the figures) shows that the upper and lower halves of the waveforms are symmetrical. This symmetry is a property of waveforms governed by any collapse rule of the form of Eq. (12) if we use the quadratic stress-strain curve, but is not generally true otherwise, as is especially evident in the example of the locking solid to be discussed subsequently.

Since steady compressive disturbances propagate as shocks in the absence of dispersive tendencies, a simple measure of the influence of this latter effect is the steady-wave rise time. Various definitions of rise time are possible, but for simplicity, ease of experimental determination, and uniform applicability to various waveforms, we define the rise time $\mathscr{T}$ as that time interval required for the strain at a fixed particle to increase from $\epsilon_0 + 0.05(\epsilon_1 - \epsilon_0)$ to $\epsilon_0 + 0.95(\epsilon_1 - \epsilon_0)$. This is the same as the corresponding value for stress or particle velocity and is given by

$$\mathscr{T} = 5.889\, T[\beta^2(\epsilon_1 - \epsilon_0)]^{-1} \tag{21}$$

for the example at hand. We see that the rise time is proportional to the ratio of the characteristic time of the material to the nonlinear correction to the wave speed, and is thus determined by the relative importance of the tendencies toward dispersion and

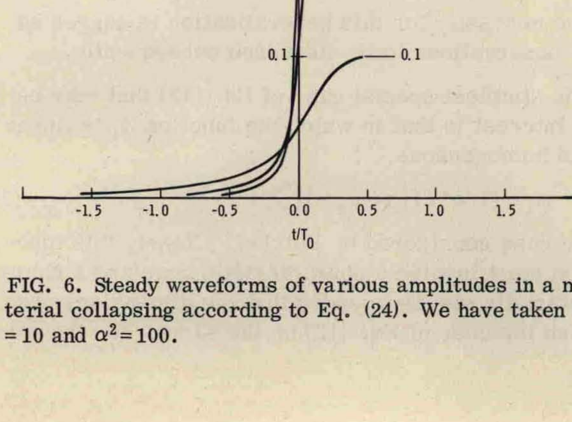

FIG. 6. Steady waveforms of various amplitudes in a material collapsing according to Eq. (24). We have taken $\beta^2 = 10$ and $\alpha^2 = 100$.

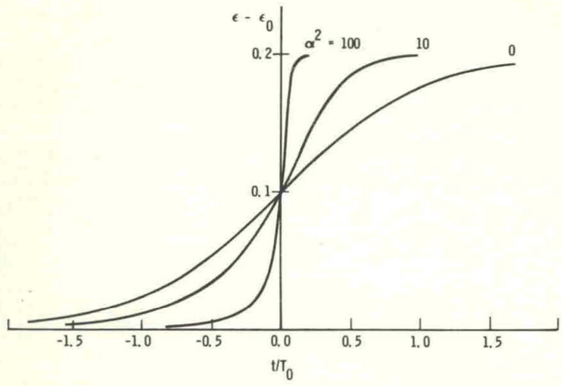

FIG. 7. Steady waveforms of amplitude $\epsilon_1 - \epsilon_0 = 0.2$ in a material collapsing according to Eq. (24). The effect of increased strain-weighting of the collapse rate is illustrated. We have taken $\beta^2 = 10$.

shock formation. Values of $\mathscr{T}/T$ given by Eq. (21) are plotted as functions of wave amplitude for various values of β^2 on Fig. 5.

B. Strain-Weighted Collapse Rule

In a recent paper, Butcher[12] examined two steady-wave profiles propagated in a rigid polyurethane foam. His results showed that the rise time was a stronger function of wave amplitude than is implied by the linear collapse model, and he expressed this discrepancy in terms of the characteristic time, saying that it seemed to be shorter for the higher amplitude wave than for the one of lower amplitude. This tends to confirm our expectation that the characteristic time for collapse should be a decreasing function of strain. To explore the consequences of such an assumption, let us consider the collapse function

$$\phi = [\sigma - \sigma_E(\epsilon)][\sigma^* T(\epsilon)]^{-1} \quad , \tag{22}$$

with $T(\epsilon)$ given by the simple relation

$$T(\epsilon) = T_0[1 + \alpha^2(\epsilon - \epsilon_0)]^{-1} \tag{23}$$

so that the collapse rate increases linearly with strain at any given overstress $\sigma - \sigma_E$.

The integral of Eq. (17) is readily evaluated for this collapse rule, and we obtain the implicit strain history

$$\frac{t}{T_0} = \frac{1}{\beta^2 \epsilon_1}\left[\log_e\left(2\frac{\epsilon}{\epsilon_1}\right) - \frac{\epsilon_1\alpha^2}{1+\epsilon_1\alpha^2}\log_e\left(2\frac{1+\epsilon_1\alpha^2(\epsilon/\epsilon_1)}{2+\alpha^2\epsilon_1}\right)\right.$$

$$\left. - \frac{1}{1+\epsilon_1\alpha^2}\log_e\left(2\,(1-\epsilon/\epsilon_1)\right)\right] \quad , \tag{24}$$

where we have taken $\epsilon_0 = 0$, $\sigma_0 = 0$, and $\sigma^* = \rho_0 c_0^2$. As before, the steady wave speed is given by Eq. (18). The rise time $\mathscr{T}$, as defined in the previous example, is found to be

$$\frac{\mathscr{T}}{T_0} = \frac{1}{\beta^2\epsilon_1(1+\alpha^2\epsilon_1)}\left[5.889 + \alpha^2\epsilon_1 \log_e\left(19\,\frac{1+0.05\alpha^2\epsilon_1}{1+0.95\alpha^2\epsilon_1}\right)\right]. \tag{25}$$

In Fig. 6 we have plotted strain histories of various amplitudes for $\alpha^2 = 100$ and $\beta^2 = 10$. To show the influence of variations of α^2 we have plotted strain histories in Fig. 7 for several choices of this parameter when ϵ_1 and β^2 are assigned the fixed values 0.2 and 10, respectively. Graphs of rise time as a function of wave amplitude are shown in Fig. 5 for $\alpha^2 = 0$ and are qualitatively the same but indicative of stronger quantitative dependence on amplitude as α^2 is increased. We note from Figs. 6 and 7 that the strengthened dependence of rise time on amplitude results primarily from a steepening of the upper portion of the wave profile.

C. Quadratic Collapse Rule

As an alternative to the introduction of strain-weighted collapse rules for increasing the dependence of rise time on amplitude we consider rules involving nonlinear dependence of the collapse rate on the overstress $\sigma - \sigma_E$. Specifically, we consider the case where

$$\phi_1 = \frac{1}{\rho_0 c_0^2 T_1}[\sigma - \sigma_E(\epsilon)]\left(1 + \frac{T_1/T_2}{\rho_0 c_0^2}[\sigma - \sigma_E(\epsilon)]\right). \tag{26}$$

It is clear that, for a given overstress, $\dot{\epsilon}$ will increase with decreasing T_2 (assuming $T_2 > 0$). When we use the function $\sigma_E(\epsilon)$ given by Eq. (11), calculation of the steady waveform is readily accomplished in the same way as before, and we arrive at the history

$$\frac{t}{T_1} = \frac{1}{\beta^2(\epsilon_1 - \epsilon_0)}\log_e\left(\frac{\epsilon - \epsilon_0}{\epsilon_1 - \epsilon}\right)$$

$$+ \frac{1}{2\gamma}\log_e\left(\frac{\gamma - [\epsilon - \frac{1}{2}(\epsilon_1 + \epsilon_0)]}{\gamma + [\epsilon - \frac{1}{2}(\epsilon_1 + \epsilon_0)]}\right), \tag{27}$$

where

$$\gamma = +\left[\frac{T_2}{T_1}\,\frac{1}{\beta^2} + \frac{1}{4}(\epsilon_1 - \epsilon_0)^2\right]^{1/2} .$$

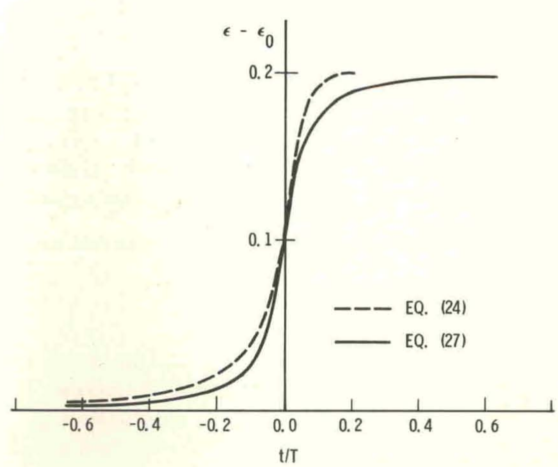

FIG. 8. Waveforms of the same amplitude and rise time calculated according to Eqs. (24) and (27). We have taken $\beta^2 = 10$, $T_0 = T_1 \equiv T$, $\alpha^2 = 100$, $T_2/T_1 = 0.00887$, and have $\tau/T = 0.4459$.

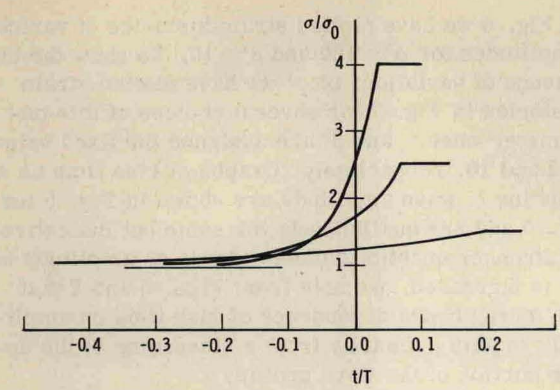

FIG. 9. Steady waveforms of various amplitudes in a locking material governed by the linear collapse rule.

Calculation of the rise time gives

$$\frac{\mathscr{T}}{T_1}=\frac{5.889}{\beta^2(\epsilon_1-\epsilon_0)}+\frac{1}{\beta^2 r}\log_e\frac{r-0.45\,(\epsilon_1-\epsilon_0)}{r+0.45\,(\epsilon_1-\epsilon_0)}\quad .$$

In the limit as $T_2\rightarrow\infty$ the waves governed by these equations become the same as those arising directly from the linear collapse rule. For finite values of T_2 the waveforms arising from the quadratic collapse theory show stronger dependence of rise time on amplitude while maintaining their symmetry.

As an indication of the sort of tailoring of waveforms that can be achieved by variations of the collapse rule, we have shown in Fig. 8 a waveform arising from the strain-weighted theory and one from the quadratic collapse theory. The equilibrium response curve and the basic time constants were chosen to be the same in each case and the parameters α^2 and T_2/T_1 were selected so that the rise time would be the same as well.

D. Locking Model

To get an indication of the influence of the equilibrium curve $\sigma_E(\epsilon)$ on steady-wave profiles, we consider wave propagation in a material governed by the strain-weighted collapse model of Eq. (22) and the locking equilibrium curve shown in Fig. 2. The locking model is of interest because it represents, in an exaggerated form, an aspect of the compaction behavior of many porous materials that is not well represented by the quadratic stress-strain curve.

The steady wave speed in a locking material is given by

$$\rho_0 V^2=(\sigma_1-\sigma_0)(\epsilon_s-\epsilon_0)^{-1}\quad .$$

Strain waves in a locking material all have the same amplitude, so we focus our attention on stress waves. Since $\sigma_E(\epsilon)=\sigma_0$ (for $\epsilon<\epsilon_s$), calculation of stress waveforms is particularly easy, and we obtain

$$t=\frac{\sigma^*(\epsilon_s-\epsilon_0)}{\sigma_1-\sigma_0}\int_{(\epsilon_s-\epsilon_0)/2}^{(\sigma-\sigma_0)(\epsilon_s-\epsilon_0)/(\sigma_1-\sigma_0)}\frac{T(\lambda)}{\lambda}\,d\lambda\,. \qquad (28)$$

The rise time, defined as before, is given by

$$\mathscr{T}=\frac{\sigma^*(\epsilon_s-\epsilon_0)}{\sigma_1-\sigma_0}\int_{0.05(\epsilon_s-\epsilon_0)}^{0.95(\epsilon_s-\epsilon_0)}\frac{T(\lambda)}{\lambda}\,d\lambda\quad .$$

In the particular case where $T(\epsilon-\epsilon_0)=T_0[1+\alpha^2\times(\epsilon-\epsilon_0)]^{-1}$, evaluation of the above integrals gives

$$t=\frac{T_0\sigma_0(\epsilon_s-\epsilon_0)}{\sigma_1-\sigma_0}\times\log_e\left(\frac{\sigma-\sigma_0}{\sigma_1-\sigma_0}\,\frac{2+\alpha^2(\epsilon_s-\epsilon_0)}{1+\alpha^2(\epsilon_s-\epsilon_0)(\sigma-\sigma_0)(\sigma_1-\sigma_0)^{-1}}\right) \qquad (29)$$

and

$$(\sigma_1-\sigma_0)\mathscr{T}=T_0\left[\sigma_0(\epsilon_s-\epsilon_0)\log_e\left(19\frac{1+0.05\alpha^2(\epsilon_s-\epsilon_0)}{1+0.95\alpha^2(\epsilon_s-\epsilon_0)}\right)\right]. \qquad (30)$$

In Fig. 9 we have shown some typical waveforms given by the special case of Eq. (29) in which $\alpha^2=0$, i.e., when collapse rule is linear, with the locking equilibrium curve. We see that these waveforms are very unsymmetrical, as one would expect from the fact that $\sigma-\sigma_E$ is largest at the peak amplitude of each wave. These waves also terminate abruptly at their peak amplitude because of the finite collapse rate there. Otherwise, their behavior is similar to that of the waves discussed previously. When we take $\alpha\neq 0$ the upper part of the waveform rises more steeply than in the case shown in Fig. 9.

V. EXPERIMENTAL DETERMINATION OF MATERIAL CONSTITUTION

In this section we discuss the inference of equilibrium stress-strain curves and collapse rules from experimental records of steady waveforms. For this purpose, we assume that one can experimentally establish the existence, form, and propagation velocity of such waves.

As a result of our discussion of this section it will become clear that, in each material, all steady waves associated with a given initial state (σ_0,ϵ_0) can be exactly reproduced by the present theory and that, for this reason, checks of the theory against experimentally determined steady waveforms give no information about its validity in the context of more general collapse processes.

A. Equilibrium Stress-Strain Curve

The equilibrium stress-strain curve is determined by applying Eq. (9) to the state behind each member of a family of steady waves of various amplitudes and measured velocities. If, for example, we have a particle-velocity history $u(t)$, u_1 is obtained as the limiting value of $u(t)$ as t becomes large. Equations (9) then give $\epsilon_1=\epsilon_0+[(u_1-u_0)/V]$ and $\sigma_1=\sigma_0+\rho_0(u_1-u_0)V$. This gives a point (σ_1,ϵ_1), and the locus of all such points, one obtained from each

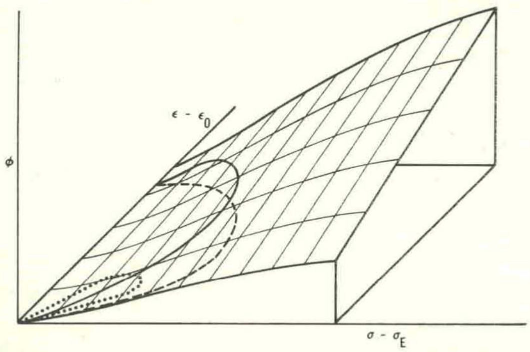

FIG. 10. Typical compaction surface $\dot{\epsilon} = \phi(\epsilon, \sigma - \sigma_E)$. The path on the surface shown as a heavy solid line is the locus of $(\epsilon, \sigma - \sigma_E, \dot{\epsilon})$ states taken at a material point during the passage of a wave. The dashed line is the projection of the path into the $(\epsilon, \sigma - \sigma_E)$ plane, and the small dotted loop is its projection into the $(\sigma - \sigma_E, \dot{\epsilon})$ plane.

recorded waveform, is the equilibrium stress-strain curve.

B. Collapse Rule

Inference of the function ϕ of Eq. (13) from a sequence of steady waveforms of various amplitudes and speeds is, in principle, a straightforward matter. Let us suppose that the data consist of several particle-velocity records, along with a measurement of V for each wave. The equilibrium curve is found as described above and can now be considered known. Using Eqs. (9), the known values of ρ_0 and V, and the known initial conditions, the stress and strain histories can be determined. Let us consider a given time t^*. From the stress history we read off $\sigma(t^*)$, and from the strain history $\epsilon(t^*)$ and $\dot{\epsilon}(t^*)$ can be determined. Since the function $\sigma_E(\epsilon)$ is known, we have values of $\dot{\epsilon}$, σ, and $\sigma - \sigma_E$ at $t = t^*$ and are thus able to plot a point of the surface ϕ as shown on Fig. 10. This process is repeated for a number of values t^* for each of the records in hand to map out a portion of the surface. From each experimental record we obtain values of ϕ associated with $(\epsilon, \sigma - \sigma_E)$ points lying on a curve in this plane (such as that shown by the dashed line in Fig. 10) that passes through the values ϵ_0 and ϵ_1 on the ϵ axis, and is single valued in ϵ. As an example we note that, in the case of a material governed by the linear collapse rule and the quadratic equilibrium behavior, these curves are parabolas with their maximum value, $(\sigma - \sigma_E)_{\max} = \rho_0 c_0^2 \beta^2 [\frac{1}{2}(\epsilon_1 - \epsilon_0)]^2$, taken at $\epsilon = \frac{1}{2}(\epsilon_1 + \epsilon_0)$.

Families of steady waveforms obtained by means of the usual plate-impact experiments have the same initial states (σ_0, ϵ_0), but different amplitudes. Since the Rayleigh lines corresponding to these waves do not cross, no two waves will have an $(\epsilon, \sigma - \sigma_E)$ point in common, so no conflicting values of ϕ can arise. By the same token, we see that the process described will always lead to a function ϕ that reproduces all of the observed waveforms exactly.

The possibilities for fitting less general collapse rules such as that of Eq. (12) or the specific forms in the examples of Sec. IV to experimental data are, of course, more limited. If the surface $\phi(\epsilon, \sigma - \sigma_E)$ determined by the means discussed above turns out to be a cylindrical sheet (in which case the generators will be parallel to the ϵ axis since this line is on the surface), then, of course, Eq. (12) is appropriate and the material is characterized by a curve in the $(\sigma - \sigma_E, \dot{\epsilon})$ plane.

As a practical matter, it may be desirable to restrict one's effort to fitting a simple collapse rule to available data. A collapse rule of the form of Eq. (12) is completely determined (over the range in question) by the highest-amplitude waveform in hand; it is just the locus of points $(\sigma - \sigma_E, \dot{\epsilon})$ obtained from this record. For most materials (idealized locking materials being the exception) $\sigma - \sigma_E$, and hence $\dot{\epsilon}$, vanish at both initial and final strains in the wave profile, and for this reason the locus of the $(\sigma - \sigma_E, \dot{\epsilon})$ points will form a closed path in this plane that begins and ends at the origin. An example of such a path is shown as the small dotted loop on Fig. 10. If the collapse rule of Eq. (12) is appropriate, the path will be a single line that is retraced for the upper portion of the wave profile. If, however, the path is a wide loop, a strong strain dependence is indicated and Eq. (12) does not provide an adequate model of the behavior. A less abstract check on the adequacy of the form is obtained by simply calculating lower-amplitude wave profiles and comparing them with experimental records. Collapse rules of such simple forms as those of Eqs. (14) and (26) can be established by choosing the coefficients T or T_1 and T_2, respectively, for best fit to the $(\sigma - \sigma_H, \dot{\epsilon})$ curve.

When the equilibrium response of the material is adequately represented by the locking model, steady-wave solutions are particularly simple and somewhat stronger statements can be made. For example, we see from Eq. (28) that a necessary condition for the applicability of the collapse rule of Eq. (22) is that the product $(\sigma_1 - \sigma_0)\mathscr{T}$ be the same for each member of a family of wave profiles. Let us suppose that this is true in some instance, and that we would like to fit the waveforms of Eq. (29) to the data. Since we have determined the constant value of the quantity $(\sigma_1 - \sigma_0)$, Eq. (30) becomes a relationship giving a one-parameter family of coefficient pairs $(T_0(\alpha^2), \alpha^2)$ for which all the waveforms given by Eq. (30) have the rise times observed in the experiments.

The remaining parameter α^2 can be adjusted to improve the agreement between the calculated and observed waveforms. As α^2 is increased the upper portion of the waveform is steepened with the lower portion being spread more to keep the rise time the same. When an approximate fit of a simple collapse rule to experimental observations is desired, the first thing to be decided upon is an appropriate criterion of goodness of fit. Usually one would like a reasonable fit to a range of waveforms rather than a perfect fit to some and a large error for others. For this purpose, one of the more reasonable criteria of good fit would be agreement between theory and experiment on the amplitude dependence of rise time.

VI. SUMMARY

In the previous sections of this paper, we have presented and discussed a simple theory of the dynamic compaction of porous solids. This theory elaborates the conventional theory of shock propagation in such a way that the observed shock structures can be described. It is not the only reasonable theory for this purpose, but does seem representative of several that could be proposed. We have shown that, for each material, the theory can be fit exactly to all steady-wave profiles having a given initial state. The same is true of several other theories in which the collapse rule involves two independent variables. Theories involving collapse rules that are special cases of Eq. (13) can, in general, be fit only approximately to experimental observations. A brief discussion of how this fitting could be accomplished has been presented. Examination of a variety of solutions such as were given in Sec. IV is helpful in deciding on the form of a collapse rule appropriate to fitting a specific set of data.

The conclusion that the collapse rule of Eq. (13) could be fit exactly to all steady-wave data following from the usual plate-impact experiment leaves open the question of how one can obtain a meaningful check of theory against experiment. From an examination of the discussion of Sec. V we see that what is needed is an experiment involving a compaction path that intersects the pencil of Rayleigh lines of the impact experiments. The two sorts of experiments that come to mind are those involving precompressed samples (so that the initial conditions are changed) and those in which evolving waves are studied. The former are reasonable and simple to perform. The latter also seem promising, but are more difficult since the theoretical predictions to be compared with the experiments must follow from solutions of the partial differential Eq. (5), along with appropriate constitutive equations.

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